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General solutions for the field of a charged particle in Brans–Dicke theory of gravitation

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Abstract. The field equations in the Brans–Dicke (BD) scalar–tensor theory of gravitation are solved for a spherically symmetric metric. The solutions generalise earlier conformally flat results and may all be considered as describing the field of a charged mass point surrounded by a scalar–tensor field. The conformally flat solutions are shown to be not physically meaningful for 'standard' BD theory with $2\omega+3>0$.

1. Introduction

This paper is intended as a generalisation of the work of Reddy (1979) and Reddy and Rao (1981) on the Brans–Dicke theory of gravity. They obtained spherically symmetric, static and conformally flat solutions of the BD vacuum and electrovacuum field equations.

The general spherically symmetric vacuum solutions were obtained by the author (Van den Bergh 1980) and resulted also in general solutions for Nordtvedt's theory of gravitation (Van den Bergh 1982). Now the general solutions of the static spherically symmetric field equations are presented for an energy–momentum tensor due to a source-free electromagnetic field, without restricting the metric to be conformally flat (Reddy and Rao 1981). In § 2, we reduce the BD–Maxwell equations to some elegant form and in § 3 the general solutions are presented and discussed. It is shown that the earlier conformally flat solutions are not physically meaningful for a standard BD scalar field with positive energy density. Section 4 contains some concluding remarks.

2. Field equations

We follow the notation and sign conventions of Reddy and Rao (1981). The BD–Maxwell field equations are

$$R_{ij}-\tfrac{1}{2}Rg_{ij}=8\pi\phi^{-1}T_{ij}+\omega\phi^{-2}(\phi_{,i}\phi_{,j}-\tfrac{1}{2}g_{ij}\phi_{,k}\phi^{,k})+\phi^{-1}\phi_{,i;j} \tag{1}$$

$$\phi^{,k}{}_{;k}=0 \qquad (\omega\neq-\tfrac{3}{2}) \tag{2}$$

$$F^{ij}{}_{;j}=0 \tag{3}$$

with

$$T_{ij}=F_{il}F^{l}{}_{j}-\tfrac{1}{4}g_{ij}F_{lm}F^{lm} \tag{4}$$

$$F_{ij}=A_{[j,i]}. \tag{5}$$

We assume now that the space–time geometry is given by the general static and spherically symmetric metric

$$ds^2 = -b\, dt^2 + a\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\Phi^2 \tag{6}$$

with a and b functions of r alone. Taking the electromagnetic field to be purely electrostatic,

$$A_i = \delta_i^0 V \tag{7}$$

with $V = V(r)$, the non-vanishing components of the energy–momentum tensor are found to be

$$T_0^0 = T_1^1 = -T_2^2 = -T_3^3 = -(2ab)^{-1}(dV/dr)^2 \tag{8}$$

with, from (3),

$$dV/dr = qr^{-2}(ab)^{1/2}. \tag{9}$$

Introducing Reddy and Rao's (1981) radial coordinate

$$\tilde{r} = rb^{-1/2} \tag{10}$$

it can be seen that one recovers their formula (7) with

$$a^{1/2} = b^{1/2}\, d\tilde{r}/dr. \tag{11}$$

It should be stressed (Will 1974) that the asymptotic value ϕ_0 of the scalar field is related to the gravitational constant G_0, as measured at infinity, by

$$\phi_0 = G_0^{-1}(4+2\omega)/(3+2\omega). \tag{12}$$

With the Brans and Dicke (1961) transformation

$$R = \phi^{1/2}\phi_0^{-1/2} r \tag{13}$$

$$B(R) = \phi\phi_0^{-1} b \tag{14}$$

$$A(R) = \phi\phi_0^{-1} a (dr/dR)^2 \tag{15}$$

the field equations now read (a superscript prime indicating differentiation with respect to R)

$$\frac{B''}{2B} - \frac{B'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{RA} = \frac{v'^2}{B} - \varphi'^2 \tag{16}$$

$$-\frac{B''}{2A} + \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{RA} = -\frac{v'^2}{A} \tag{17}$$

$$-1 + \frac{R}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A} = -\frac{R^2 v'^2}{AB} \tag{18}$$

$$[R^2(B/A)^{1/2}\varphi']' = 0 \tag{19}$$

where we have defined

$$v = (4\pi)^{1/2}\phi_0^{-1} V \tag{20}$$

$$\varphi = [\tfrac{1}{2}(2\omega + 3)]^{1/2} \ln(\phi\phi_0^{-1}). \tag{21}$$

Only scalar fields with positive energy density $(2\omega+3>0)$ will be considered. Defining

$$\rho=(4\pi)^{1/2}q\phi_0^{-1}R^{-1} \tag{22}$$

(17), (19) and (20) yield respectively

$$\mathrm{d}^2B/\mathrm{d}v^2=0 \tag{23}$$

$$(\mathrm{d}/\mathrm{d}v)(B\,\mathrm{d}\varphi/\mathrm{d}v)=0 \tag{24}$$

$$\mathrm{d}^2\rho/\mathrm{d}v^2=\tfrac{1}{2}\rho(\mathrm{d}\varphi/\mathrm{d}v)^2 \tag{25}$$

whereas A follows from (9), or

$$A=B^{-1}(\mathrm{d}\rho/\mathrm{d}v)^{-2}. \tag{26}$$

In § 3 the solutions for this system will be explicitly given, under the boundary conditions

$$A=B=1 \qquad (\text{or } a=b=1) \tag{27}$$

and

$$\varphi=v=0 \tag{28}$$

at infinity.

3. Solutions of the field equations

We assume $q>0$, hence $v\leqslant 0$ and

$$\rho=-v+\mathrm{O}(v^2) \tag{29}$$

at infinity. Then (23) yields

$$B=1+2\lambda v+v^2 \tag{30}$$

with λ an integration constant (taken to be positive, in order to have positive-mass solutions). From (24) one has then

$$\varphi=\mu\int_0^v(1+2\lambda x+x^2)^{-1}\,\mathrm{d}x \tag{31}$$

with μ a second integration constant.

(*a*) $\lambda=1$.
Then (31) gives

$$\varphi=\mu v(1+v)^{-1} \tag{32}$$

and the unique solution of (25) compatible with (27)–(29) is

$$\rho=-(\sqrt{2}/\mu)(1+v)\sinh[\mu v/\sqrt{2}(v+1)]. \tag{33}$$

Hence

$$A=\left(\operatorname{sech}^2\frac{\mu v}{\sqrt{2}(v+1)}\right)\left(1+\frac{\sqrt{2}}{\mu}(1+v)\tanh\frac{\mu v}{\sqrt{2}(v+1)}\right)^{-2}. \tag{34}$$

(*b*) $\lambda < 1$.
Define $\gamma = \cos^{-1}\lambda$ and

$$X = \tan^{-1}(\cot\gamma + v \operatorname{cosec}\gamma). \tag{35}$$

Then (30) and (31) yield with $-\pi/2 < X \leqslant \pi/2 - \gamma$

$$B = (\sin^2\gamma)\sec^2 X \tag{36}$$

$$\varphi = \mu(\operatorname{cosec}\gamma)(X + \gamma - \pi/2). \tag{37}$$

If $\frac{1}{2}\mu^2 \operatorname{cosec}^2\gamma < 1$ one takes $\beta^2 = 1 - \frac{1}{2}\operatorname{cosec}^2\gamma$ and one verifies that the unique solution of (25), (27)–(29) is

$$\rho = \beta^{-1}(\sec X)\sin[\beta(\pi/2 - \gamma - X)] \tag{38}$$

and hence

$$A = [\sec^2\beta(\pi/2 - \gamma - X)]\{1 - \beta^{-1}(\tan X)\tan[\beta(\pi/2 - \gamma - X)]\}^{-2}. \tag{39}$$

When, on the other hand, $\frac{1}{2}\mu^2 \operatorname{cosec}^2\gamma \geqslant 1$, one takes $\beta^2 = \frac{1}{2}\mu^2 \operatorname{cosec}^2\gamma - 1$ and then

$$\rho = \beta^{-1}(\sec X)\sinh[\beta(\pi/2 - \gamma - X)] \tag{40}$$

$$A = \{\operatorname{sech}^2\beta(\pi/2 - \gamma - X)\}\{1 - \beta^{-1}(\tan X)\tanh[\beta(\pi/2 - \gamma - X)]\}^{-2}. \tag{41}$$

(*c*) $\lambda > 1$.
Define $\gamma = \cosh^{-1}\lambda$ and

$$X = \frac{1}{2}\ln\frac{\coth\gamma + v \operatorname{cosech}\gamma - 1}{\coth\gamma + v \operatorname{cosech}\gamma + 1}. \tag{42}$$

Then (30) and (31) yield with $X \in]-\infty, -\gamma]$

$$B = (\sinh^2\gamma)\operatorname{cosech}^2 X \tag{43}$$

$$\varphi = \mu(\operatorname{cosech}\gamma)(X + \gamma). \tag{44}$$

Taking $\beta^2 = 1 + \frac{1}{2}\mu^2 \operatorname{cosech}^2\gamma$, the unique solution of (25), (27)–(29) is now

$$\rho = -\beta^{-1}(\operatorname{cosech} X)\sinh[\beta(X + \gamma)] \tag{45}$$

and hence

$$A = [\operatorname{sech}^2\beta(X + \gamma)]\{1 - \beta^{-1}(\coth X)\tanh[\beta(X + \gamma)]\}^{-2}. \tag{46}$$

By some tedious but straightforward calculations one can verify that in all solutions presented here, A, B and ρ are smooth and strictly positive on the domain of definition of v, i.e. $]-1, 0[$, $]-\infty, 0[$ and $]\sinh\gamma - \cosh\gamma, 0[$ for the cases (a), (b) and (c) respectively. Furthermore, the Ricci scalar $\mathscr{R}$ can be shown to be

$$\mathscr{R} \sim \rho^4 B^{-1} \tag{47}$$

near $R = 0$, and hence all solutions have a curvature singularity at $R = 0$ and no event horizon (as $B > 0$ for $R \neq 0$).

Of course, this singularity does not have to be real, since, in a realistic situation, the BD electrovac solutions have to be matched to a charged interior solution. To the author's knowledge, no such attempt has yet been made for BD theory.

We now have a look at the conformally flat solutions of Reddy and Rao (1981). The condition for (6) to be conformally flat reads

$$a = [1 - \tfrac{1}{2}r(\mathrm{d}/\mathrm{d}r)\ln b]^2 \tag{48}$$

or, by the use of (13)–(15), (22) and (30)

$$\mathrm{d}\rho/\mathrm{d}v = -B^{-1}\rho(\lambda + v) + B^{-1/2} \tag{49}$$

and hence

$$\rho = (\rho_0 - v)B^{-1/2} \tag{50}$$

with ρ_0 an integration constant.

Asymptotical flatness of the solutions implies $\rho_0 = 0$, and hence (30), (31) and (50) yield

$$\mu^2 \equiv -2v^{-1}[2\lambda v^2 + (\lambda^2 + 3)v + 2\lambda] \tag{51}$$

which is clearly impossible for real values of λ. It follows that conformally flat solutions of the BD–Maxwell equations (with $\omega > 0$) do not exist!

What then does one make of the Reddy and Rao solutions (1981), which effectively are conformally flat? A careful look at equation (14) of their paper shows that asymptotical flatness of the solution implies

$$\phi_0 < 0. \tag{52}$$

But this means that the *effective* gravitational constant G_0, as measured at infinity, would be negative. This can be cured by taking the same solution, but with $2\omega + 3 < 0$. Indeed, replacing (21) by

$$\varphi = |\tfrac{1}{2}(2\omega + 3)|^{1/2} \ln \phi\phi_0^{-1} \tag{53}$$

one has then

$$\mathrm{d}^2\rho/\mathrm{d}v^2 = -\tfrac{1}{2}\rho(\mathrm{d}\varphi/\mathrm{d}v)^2 \tag{54}$$

instead of (25). This causes the RHS of (51) to be positive for $\lambda = 0$. The solutions are then given by (36)–(39), but now with $\beta^2 - 1 = \frac{1}{2}\mu^2 \operatorname{cosec}^2 \gamma = \frac{1}{2}\mu^2$. Reddy and Rao's (1981) solution corresponds to $\beta = 2$ or $\mu = 6^{1/2}$, yielding $v = \tan X$, $\rho = -\sin X$ and $B = \sec^2 X$.

4. Conclusion

We obtained explicit solutions for the static and spherically symmetric coupled BD–Maxwell equations, representing a charged mass point surrounded by a scalar–tensor field with $2\omega + 3 > 0$. The solutions have a naked time-like singularity at the origin. This property remains true, even in the limit of vanishing coupling of the scalar field with the gravitational field ($\omega \to \infty$)! Accidentally this shows that, although the field equations of BD theory approach those of general relativity for large values of ω, the global properties of the solutions (e.g. the ones presented here, and the Reissner–Nordström solution) stay quite distinct. Herewith we generalise an earlier result, obtained by Janis *et al* (1968) for uncharged fields.

Conformally flat and asymptotically flat solutions, as predicted in earlier work, are shown to be non-existing for $2\omega + 3 > 0$.

The unique known conformally flat solution corresponds in fact to a negative energy density scalar field ($2\omega + 3 < 0$). This solution is singular too. However, other negative energy density scalar–tensor solutions may turn out to be useful, e.g. in particle physics (Clément 1981).

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